On the fractional derivative of stationary stochastic processes

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Abstract

The problem of the definition of a fractional derivative suitable for stationary stochastic processes is faced in this paper. To do it two sets of fractional derivatives are considered: a) the forward and backward and b) the central derivatives. It is shown that for these derivatives the corresponding autocorrelation functions have the same representations. The obtained results are used to define a fractional noise and, from it, the fractional Brownian motion. This is studied and it is shown that the proposed formulation leads to Mandelbrot's definition.

Keywords: forward and backward fractional derivatives; generalised Cauchy derivative; Liouville derivative; differintegration; central fractional derivatives; fractional stochastic process; fractional Brownian motion.

1. Introduction

Stochastic processes with fractional characteristics are very frequent in Nature and in daily applications. Fractional Brownian motion (fBm) and 1/f noises are well known designations for some of these kinds of signals [1-4]. In parallel, self-similarity and long range dependence are interconnected notions and appear in a variety of contexts [1-4]. However, it is not clear how we can establish a bridge between these notions and the current definitions of fractional derivative. Here we will try to present a new step into that goal.

Starting from several fractional derivative definitions, we apply them to stationary stochastic processes. The computation of the autocorrelations of the derivative processes shows that they are equivalent. We make a particularization to the white noise and use the results to define a fractional Brownian motion. This is a model for non stationary signals, but with stationary increments, that are useful in understanding phenomena with long range dependence and with a frequency dependence of the form $1/f^{\alpha}$, with α non integer. A mathematical representation for this kind of process was proposed by Mandelbrot and Van Ness [1] and can be stated as follows.

Let H, 0<H<1, be the called Hurst parameter and b_0 a number. Then the fBm, $B_H(t)$, with parameter H is defined by:

$$B_H(t)-B_H(0) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}\right] dB(\tau) + \int_{0}^{t} (t-\tau)^{H-1/2}\, dB(\tau)\right\} \quad (1)$$

where $B_H(0) = b_0$, and B(t) is the standard Brownian motion.

Following a common practice in engineering texts we will introduce the stationary white noise process, w(t), instead of dB(t). Although there are other definitions and approaches to fBm, the above definition is usually accepted. In this paper, we will show that the above definition can be deduced from ours.

The paper outlines as follows. In section 2, we study the fractional derivatives: causal and central. They are used in section 3 to compute the autocorrelation function of the derivative processes. The proposed approach for defining fBm is presented in the following section, where we will study its main features. At last, some simulations will be presented.

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2. On the fractional derivative definitions

2.1 The forward and backward derivatives

Let f(z) be a complex variable function analytic in a region that includes a half straight line starting at z and defined by z-nh, with $n \in Z$; h is any complex in the right hand d'Argand plane. Consider the U shaped contour represented in figure 1.

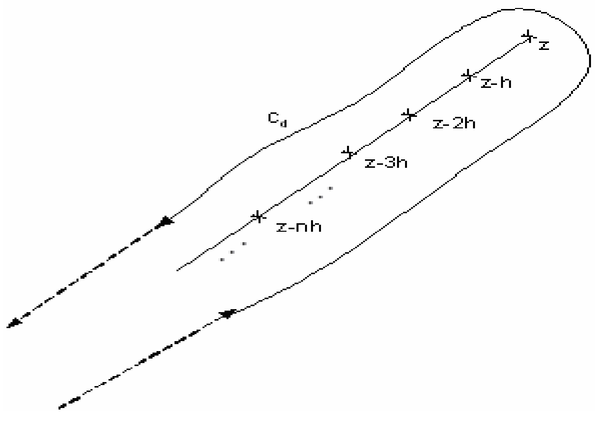

Fig. 1 – U shaped contour

Assume that this line is inside the analyticity region.

Definition 1 – We define the forward and backward Grünwald-Letnikov fractional α order derivatives by the left hand sides in (2) and (3) below.

With these definitions and under the above conditions we can state the following interesting result [5,6]:

Theorem 1 –

$$\lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha} = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_{C_d} f(w) \frac{1}{(w-z)^{\alpha+1}}\, dw \qquad (2)$$

The right hand side is the generalised Cauchy derivative. We will write $D_+^\alpha f(z)$ for representing this derivative. Making a substitution h→-h we obtain the backward Grünwald-Letnikov derivative on the left and a generalised Cauchy with a branch cut line on the right half plane

$$D_-^\alpha f(z) = \lim_{h\to 0+} (-1)^\alpha \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha}$$

$$= \frac{\Gamma(\alpha+1)}{2\pi i} \oint_{C_d} f(w) \frac{1}{(w-z)^{\alpha+1}}\, dw \qquad (3)$$

We must remark that the right hand side remains the same excepting the integration path.

We can go a bit further by deforming the contour used in (2) and (3) in order to transform it in the Hankel path [7,8]. We obtain:

Theorem 2 –

$$D_\pm^\alpha f(z) = \frac{e^{j(\pi-\theta)\alpha}}{\Gamma(-\alpha)} \int_0^\infty \frac{\left[f(x.e^{j\theta}+z) - \sum_0^N \frac{f^{(n)}(z)}{n!} e^{jn\theta} x^n \right]}{x^{\alpha+1}}\, dx \qquad (4)$$

where θ is the angle between the positive real axis and the branch cut line. This is a regularised integral "a la Hadamard", but obtained without rejecting any infinite part. If θ = π (+), we have the forward derivative, while with θ = 0 (-), we obtain the backward one.

Corollary 1 – If f(z) is a Laplace transformable function, it can be shown that [7]

$$D_+^\alpha f(z) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^\infty f(z-\tau).\tau^{-\alpha-1}\, d\tau \qquad (5)$$

and

$$D_{-}^{\alpha} f(z) = \lim_{h\to 0+} (-1)^{\alpha} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^{\alpha}} = \frac{(-1)^{-\alpha}}{\Gamma(-\alpha)} \int_0^{\infty} f(z+\tau).\tau^{-\alpha-1}\, d\tau \qquad (6)$$

where the right hand sides in (5) and (6) are the forward and backward Liouville derivatives. They correspond to the output of a fractional differintegrator [7]. In the following, we will work with the forward derivative.

2.2 The central derivatives

Assume that $\alpha > -1$ and $h \in R^{+}$[1].

Definition 2 – We define the type 1 and 2 fractional central derivatives, respectively by [9,10]:

$$D_{c_1}^{\alpha} f(t) = \lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha/2-k+1)\,\Gamma(\alpha/2+k+1)} f(t-kh) \qquad (7)$$

and

$$D_{c_2}^{\alpha} f(t) = \lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]} f(t-kh+h/2) \qquad (8)$$

Theorem 3 – In the above conditions, the integral formulation for the type 1 derivative is

$$D_{c_1}^{\alpha} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_c} f(z+w) \frac{1}{(w)_l^{\alpha/2+1}\,(-w)_r^{\alpha/2}}\, dw \qquad (9)$$

while for the type 2 derivative is

$$D_{c_2}^{\alpha} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_c} f(z+w) \frac{1}{(w)_l^{(\alpha+1)/2}\,(-w)_r^{(\alpha+1)/2}}\, dw \qquad (10)$$

The subscripts “l” and “r” mean respectively that the power functions have the left and right half real axis as branch cut lines. These integrals represent again generalisations of the Cauchy derivative.

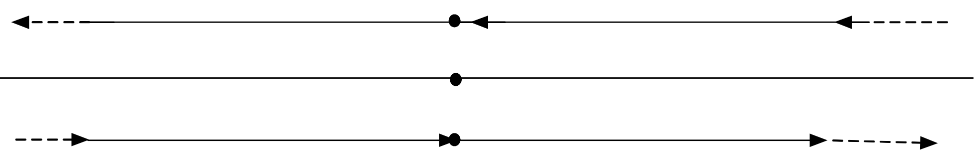

Fig. 2 – Segments for the computation of the integrals (9) and (10)

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1 Obviously that we can use complex values, but the results are similar; so, we will not do it, here.

If we had chosen an integration path making an angle θ with the real axis – it corresponds to using a complex h in (7) and (8) – we would obtain similar results.

Corollary 2 – Computing the integrals using the above paths, we obtain for the type 1 case [9,10]

$$\lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^\alpha} \sum_{-\infty}^{+\infty} \frac{(-1)k}{\Gamma(\alpha/2-k+1)\,\Gamma(\alpha/2+k+1)} f(t-kh) =$$

$$= \frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} f(z-x)\frac{1}{|x|^{\alpha+1}}\,dx \qquad (11)$$

and or the type 2 case:

$$\lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^\alpha} \sum_{-\infty}^{+\infty} \frac{(-1)k}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]} f(t-kh+h/2) =$$

$$= -\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} \int_{-\infty}^{\infty} f(z-x)\frac{\mathrm{sgn}(x)}{|x|^{\alpha+1}}\,dx \qquad (12)$$

The right hand sides in (11) and (12) are the so called Riesz and modified Riesz potentials [11]. They supply us with two integral formulations for the central derivatives that can be useful as we will see later. They have an interesting feature: one is the Hilbert transform of the other.

3. Derivatives of stationary stochastic processes

Assume that f(t) is a stationary stochastic process with $R_f(t)$ as its autocorrelation function. We will assume also that f(t) has a zero mean equal to zero. If this is not the case, we may have problems when α is negative, since the corresponding derivatives will be divergent; when α is positive the derivative will be zero, for all the definitions.

In agrement with the results in the previous section and, at least conceptually, we can use all the above formulae for defining the fractional derivative of a stationary stocahstic process. However, we must be careful because, when dealing with a stationary stochastic process, the derivative must be defined under some kind of mean. We will define it in mean square sense. To go a bit further, we will compute the autocorrelation function of the stochastic processes obtained by (2) and (6) [2].

[2] It is not hard to show that the backward derivative leads to an autocorrelation equal to the one obtained by the forward. Similarly, the type 1 and type 2 centred derivatives have equal autocorrelations.

3.1 Forward case

From (2), we can conclude that:

$$R_f^{\alpha}(t_1, t_2) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{\alpha}{k} (-1)^{k-n} \binom{\alpha}{n} R_f[t_1-t_2-(k-n)h]}{h^{2\alpha}} \tag{13}$$

With a change in the summation variable, it is not hard to show that, if $\alpha > -1/2$ [12]:

$$R_f^{\alpha}(t_1, t_2) = \lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)} R_f(t_1-t_2-kh) \tag{14}$$

So, the autocorrelation function of the forward α-order derivative of a stationary stochastic process is the type 1 centred derivative of order 2α, if $\alpha>-1/2$. This means that the forward fractional derivative of order $\leq -1/2$ is not stationary or may not exist.

With the backward derivative, we would obtain the same result. So, we will use only the forward, since it is a causal derivative.

3.2 Central case

For this case, we proceed as above, but using the type 1 derivative (7). We have then:

$$R_f^{\alpha}(t_1, t_2) =$$

$$\lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \frac{(-1)^{k+n} R_f[t_1-t_2-(k-n)h]}{\Gamma(\alpha/2-k+1)\,\Gamma(\alpha/2+k+1)\,\Gamma(\alpha/2-n+1)\,\Gamma(\alpha/2+n+1)} \tag{15}$$

We can simplify this expression with a change in the summation variable in one series, and using the following relation [13]:

$$\sum_{-\infty}^{+\infty} \frac{1}{\Gamma(a-k+1)\,\Gamma(b-k+1)\,\Gamma(c+k+1)\,\Gamma(d+k+1)} =$$

$$= \frac{\Gamma(a+b+c+d+1)}{\Gamma(a+c+1)\,\Gamma(b+c+1)\,\Gamma(a+d+1)\,\Gamma(b+d+1)} \tag{16}$$

With it, it is not difficult to show that:

$$R_f^{\alpha}(t_1, t_2) = \lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)} R_f(t_1-t_2-kh) \tag{17}$$

that is equal to (14).

With a similar procedure, we can show that the same result is obtained if one used the type 2 derivative.

3.3 Which one?

From the above results we conclude that, from the autocorrelation point of view, all the definitions lead to the same result. This conclusion was probably expected, since for a stationary stochastic process, there is no privileged direction of time flow.

These considerations mean that it seems to be indifferent to use one or other derivative. The problem at hand can give us some insight into the definition we should adopt. In problems involving time as independent variable we must use (2) because of its causal character. When causality is not involved we must use a central derivative.

3.4 Some computational issues

Looking at (14) and (17) we note that the right hand side is function of $t_1 - t_2$ only. So, we can write:

$$R_f^{\alpha}(t) = \lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)} R_f(t - kh) \qquad (18)$$

We must be careful in using it, since it may not lead to a valid autocorrelation function of a stationary stochastic process.

According to the central type 1 derivative definition and, as we referred before,

$$R_f^{\alpha}(t) = D_{c_1}^{2\alpha} R_f(t) \qquad (19)$$

and

$$R_f^{\alpha}(t) = \frac{1}{2\Gamma(-2\alpha)\cos(\alpha\pi)} \int_{-\infty}^{\infty} R_f(z-x) \frac{1}{|x|^{2\alpha+1}} dx \qquad (20)$$

We must refer that this result can also be obtained from the Liouville fractional derivative (5)[3].

In practical applications, we may need to compute a derivative of a signal for which a closed form is not available and we are obliged to truncate the summation or the integral. This leads to an error. This problem was studied in [8] in connection with the called there “short-memory” principle. Here, we will obtain a similar result can be obtained for the autocorrelation case. Consider the truncation of the integral in (20) and assume that the autocorrelation of the original process f(t) is a bounded function – $|R_f(t)|<M$ – known inside an interval that we will assume to be symmetric relatively to the origin, [-L; L], only by simplicity. We conclude that the error is bounded:

[3] So, the regularised versions are not useful for stationary stochastic processes.

$$E < \frac{M}{|\Gamma(-2\alpha)\cos(\alpha\pi)|}\int_L^\infty \frac{1}{x^{2\alpha+1}}dx = \frac{ML^{-2\alpha}}{|\Gamma(-2\alpha)\cos(\alpha\pi)|} = \frac{|\Gamma(2\alpha+1)|}{\pi}ML^{-2\alpha} \tag{21}$$

This result is identical to the one obtained in [8]. A similar result can be obtained for the summation in (18). However, here we have an error bound that is function of h. From the properties of the gamma functions, we obtain easily:

$$\frac{(-1)^k}{\Gamma(\alpha-k+1)} = -\frac{\sin(\alpha\pi)}{\pi}\Gamma(-\alpha+k) \tag{22}$$

and

$$\frac{(-1)^k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)} = -\frac{\sin(\alpha\pi)}{\pi}\frac{\Gamma(-\alpha+|k|)}{\Gamma(\alpha+|k|+1)} \tag{23}$$

As the quotient of the two Gamma functions in (23) tends asymptotically enough to $|k|^{-2\alpha-1}$ when |k| is high [11,14], we obtain

$$\left|\frac{(-1)^k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)}\right| \sim \frac{1}{\pi}|k|^{-2\alpha-1} \tag{24}$$

This leads to an error:

$$E(h) \sim \frac{|\Gamma(\alpha+1)|}{\pi}\sum_{L+1}^{+\infty}|k/h|^{-2\alpha-1}\,h \tag{25}$$

and leads to (21) again.

4. The fractional Brownian motion

Now, we are going to use the above results to study the well-known fractional Brownian motion (fBm). We will retake the approach proposed in [15].

Assume now that we are computing the fractional derivative of the white noise, w(t). We will assume it has power equal to σ^2.

Definition 3 – We define a fractional noise by:

$$r_\alpha(t) = D^\alpha w(t) \tag{26}$$

If w(t) is Gaussian, we will call $r_\alpha(t)$ fractional Gaussian noise.
As known, the autocorrelation function of the white noise is given by the Dirac delta, $\sigma^2\delta(t)$. Inserting this into (18), we obtain for the derivative autocorrelation:

$$R_r^\alpha(t) = \lim_{h\to 0}\frac{\Gamma(2\alpha+1)}{h^{2\alpha}}\sum_{-\infty}^{+\infty}\frac{(-1)k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)}\delta(t-kh) \tag{27}$$

The right hand side is a sequence of weighted impulses that become close together as h goes to zero. From (20) we conclude immediately that:

$$R_r^\alpha(t) = \frac{1}{2\Gamma(-2\alpha)\cos(\alpha\pi)}|t|^{-2\alpha-1} \tag{28}$$

giving an interesting result:

$$|t|^{-2\alpha-1} = \frac{\pi}{\sin(-\alpha\pi)} \lim_{h\to 0} \frac{1}{h^{2\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)} \delta(t - kh) \qquad (29)$$

valid for $\alpha > -1/2$. Returning back to (28) we can deduce that we must have

$$\begin{cases} 2\alpha+1>0 \\ \Gamma(-2\alpha)\cos(\alpha\pi)>0 \end{cases} \qquad (30)$$

to guarantee that (28) represents an autocorrelation function, having a maximum at the origin. The first condition ($\alpha > -1/2$) was already assumed. As

$$\frac{1}{2\Gamma(-a)\cos(a\pi/2)} = -\frac{\Gamma(a+1).\sin(a\pi)}{2\pi\cos(a\pi/2)} = -\frac{\Gamma(a+1)\sin(a\pi/2)}{\pi} \qquad (31)$$

it is not hard to see that for $-1/2 < \alpha < 0$ and $\alpha\in(2n+1,2n)$, $n\in Z^+$ we obtain valid autocorrelation functions. This fractional noise will be used next to define the fractional Brownian motion.

Definition 4 – Let $r_\alpha(t)$ be a fractional noise. Define a process $v_\alpha(t)$, $t\geq 0$, by:

$$v_\alpha(t) = \int_0^t r_\alpha(\tau)d\tau \qquad (32)$$

We will call this process a fractional Brownian motion (or generalised Wiener-Lévy process).

This process is a generalisation of the ordinary Brownian noise that is obtained with $\alpha = 0$. It is not hard to show that it enjoys all the properties normally required for the fBm [15]:

1 - $v_\alpha(0) = 0$ and $E\{v_\alpha(t)\} = 0$ for every $t\geq 0$ [4].

2 – The covariance is [15]:

$$E[v_\alpha(t)\, v_\alpha(s)] = \frac{\sigma^2}{2\Gamma(-2\alpha+2).\cos\alpha\pi} \left[|t|^{-2\alpha+1}+|s|^{-2\alpha+1} - |t - s|^{-2\alpha+1}\right] \qquad (33)$$

Putting $H= - \alpha + 1/2$ wth $H\in]0,1[$, then $\alpha\in]-1/2,1/2[$ and we obtain the usual formulation:

$$E[v_\alpha(t)\, v_\alpha(s)] = \frac{V_H}{2}\left[|t|^{2H}+|s|^{2H} - |t - s|^{2H}\right] \qquad (34)$$

with

$$V_H= \frac{\sigma^2}{\Gamma(2H+1)\sin H\pi} \qquad (35)$$

a much more simple expression than the usual. The variance is readily obtained:

$$E[v_\alpha(t)^2] = V_H\, |t|^{2H} \qquad (36)$$

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[4] If w(t) is Gaussian, so it is $r_\alpha(t)$ and $v_\alpha(t)$. The proposed definitions do not need the gaussianity.

3 – The process has stationary increments.
Letting the increments be defined by

$$\Delta v_\alpha(t,s)= v_\alpha(t) - v_\alpha(s) = \int_s^t r_\alpha(\tau)d\tau \qquad (37)$$

its variance is given by [15]:

$$\mathrm{Var}\left\{\Delta v_\alpha (t,s)\right\} = \sigma^2 \frac{|t-s|^{-2\alpha+1}}{2\Gamma(-2\alpha+2).\cos\alpha\pi} \qquad (38)$$

4 – The process is self similar
From (34), we have:

$$E[v_\alpha(at)\, v_\alpha(as)] = \frac{V_H}{2}[|a.t|^{2H}+|a.s|^{2H} - |a.t - a.s|^{2H}]=$$

$$= \frac{V_H}{2} |a|^{2H} [|t|^{2H}+|s|^{2H} - |t - s|^{2H}] \qquad (39)$$

5 – The incremental process has a $1/f^\beta$ spectrum
Defining an incremental process by (37) and choosing s = t-T:

$$d_H(t)= v_H(t) – v_H(t-T) \qquad (40)$$

has an autocorrelation function given by

$$R_d(\tau) = \frac{V_H}{2} [|\tau+T|^{2H} + |\tau-T|^{2H}- 2|\tau|^{2H}] \qquad (41)$$

and, as [10]

$$FT\left[\frac{1}{2\Gamma(\beta)\cos(\beta\pi/2)} |t|^{\beta-1}\right] = \frac{1}{|\omega|^\beta} \qquad (42)$$

we obtain the spectrum of the incremental process:

$$S_d(\omega) = \sigma^2.\frac{\sin^2(\omega T/2)}{|\omega|^{2H+1}} \qquad (43)$$

For $|\omega| << \pi/T$, the spectrum can be approximated by:

$$S_d(\omega) \approx.\frac{\sigma^2 T^2}{4}\frac{1}{|\omega|^{2H-1}} \qquad (44)$$

We conclude that the proposed definition agrees with Mandelbrot and van Ness results.
The result expressed in (44) is interesting [1,2]:

a) If 0<H<1/2, the spectrum is parabolic and corresponds to an antipersistent fBm, because the increments tend to have opposite signs;
b) If 1/2<H<1, the spectrum has a hyperbolic character and corresponds to a persistent fBm, because the increments tend to have the same sign.

We are going to show that our (32) and Mandelbrot and van Ness definitions are formally equivalent, although ours is more general and allows other possibilities in which concerns to practical implementation.

Let us return to (1), rewrite it in the format:

$$\frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0} w(\tau)\left[(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}\right] d\tau + \int_{0}^{t} w(\tau)(t-\tau)^{H-1/2}\, d\tau\right\} =$$

$$= \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{+\infty} w(\tau)\left[(t-\tau)^{H-1/2}u(t-\tau) - (-\tau)^{H-1/2}u(-\tau)\right]\right\} d\tau$$

where u(t) is the Heaviside unit step. The inner expression can be considered as the application of the Barrow formula to the function $(t-\tau)^{H-1/2}u(t-\tau)$. So, there a function $(t-\tau)^{H-3/2}u(t-\tau)$, such that:

$$\frac{1}{\Gamma(H-1/2)}\int_{0}^{t} (s-\tau)^{H-3/2}u(s-\tau)\, ds = \frac{1}{\Gamma(H+1/2)}\left[(t-\tau)^{H-1/2}u(t-\tau) - (-\tau)^{H-1/2}u(-\tau)\right]$$

With this, we have:

$$B_H(t)-B_H(0) = \frac{1}{\Gamma(H-1/2)}\int_{-\infty}^{+\infty} w(\tau)\int_{0}^{t} (s-\tau)^{H-3/2}u(s-\tau)\, ds\, d\tau$$

$$= \frac{1}{\Gamma(H-1/2)}\int_{0}^{t}\int_{-\infty}^{s} w(\tau)(s-\tau)^{H-3/2}\, d\tau\, ds \qquad (45)$$

The inner integral is the forward derivative of order H-1/2. This means that

$$B_H(t)-B_H(0) = v_\alpha(t) \qquad (46)$$

for $-1/2 < \alpha < 1/2$.

We can conclude that the definition (1) is a special case of the one proposed here, but with the inconvenient: it uses the difference of two outputs of an unstable differintegrator [15].

The formulation we proposed here is more general in the sense of giving the possibility of using other derivative, specially the Grünwald-Letnikov derivative that can be useful in discrete-time implementation. In fact, assume that a small sampling interval, T = h, is used and that we use the trapezoidal rule for the integral in (32). Let $v_\alpha(n) = v_\alpha(t)$, for t = nT. We have:

$$v_\alpha(n) = \sum_{1}^{n} \frac{r_\alpha(k-1)+ r_\alpha(k)}{2}\, T \qquad (47)$$

with

$$r_\alpha(n) = \frac{1}{T^\alpha}\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} w(n-k) \qquad (48)$$

that allows us to insert into (47) and write:

$$v_\alpha(n) = \frac{1}{T^{\alpha-1}} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} \sum_{1-m}^{n-m} \frac{w(j-1)+ w(j)}{2} \qquad (49)$$

As $1/T^{\alpha-1}$ is merely a scale parameter, we can drop it and write

$$v_H(n) = \sum_{m=0}^{\infty} (-1)^m \binom{-H+1/2}{m} \sum_{1-m}^{n-m} \frac{w(j-1)+ w(j)}{2} \qquad (50)$$

that allows us to generate a fBm in two steps:

a) Smooth the white noise by adding the values inside a n length window
b) Compute the GrünwaldLetnikov of the smoothed noise.

The increments of the fBm are also easily generated.

5. Conclusions

In this paper, we made a brief approach into the definition of a fractional derivative suitable for stationary stochastic processes. To do it we considered two sets of fractional derivatives: a) the forward and backward and b) the central derivatives. For both sets we presented two formulations: a) Grünwald-Letnikov and similar, and b) Integral. We showed that for these derivatives the corresponding autocorrelation functions have the same representations. The results we obtained were used to define a fractional noise and, from it, the fractional Brownian motion. This was studied and we confirmed its properties. We showed that the proposed formulation leads to Mandelbrot's definition.

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