Business Management

Financial Calculus

1st Semester

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The Time Value of Money

- **Time Value of Money:**

  i) today: the lower the prices,
  the more Money can buy

  ii) in time: interest rate (opportunity cost) $r+\prod$
  (real interest rate + inflation)

- Notation:

  V_0 – Value of capital invested in moment 0.

  V_n – Value of capital in a future moment n.

  n – number of time units between the initial moment 0 and the present moment

  i – interest rate per period of time.

Capitalization

- **Compound interest regime:** Financial Application that yields interest payment on the initial capital AND on previous earned interest.

V_0 *initial capital (period 0)*

$V_1=V_0(1+i)$ *is the accumulated capital after one period of time*

$V_2= V_1 (1+i) = [V_0 (1+i)] (1+i) = V_0 (1+i)^2$ *is the accumulated capital after two periods of time*

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$$V_n = V_0 (1+i)^n$$

Accumulated value after n periods of time (under a compound interest regime)

Capitalization

- **Simple interest regime:** Financial application that yields interest payments on the initial capital only.

Ex.

$V_1 = V_0 (1+i)$

$V_2 = V_1 + V_0 i = V_0 + 2 V_0 i = V_0 (1+2i)$

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$$V_n = V_0 (1 + n i)$$

Accumulated value after n periods of time (under a simple interest regime)

Capitalization

- ***Capitalization (def):***

  Result of a financial application of capital over a given period of time during which interest is earned.

  - *Final or Future Value:* Total value in a future moment.

    $V_t = V_0(1+i)^t$ *Future value (in moment t) under a compound interest regime*

Discounting

- ***DIscounting (def):***

  Amount today that is equivalente do a certain value in a future moment (knowing that interest will be earned between today and the future moment).

  - *Present Value:* The value today of a future amount to be paid/received.

    Ex. $V_0 = V_t(1+i)^{-t}$

    or $V_0 = \frac{V_t}{(1+i)^t}$

Interest Rates

- ***Proporcional interest rate***: The interest rate that results from dividing the annual interest rate by the number of periods existent in one year.
  - → i periodic: i_k ;
  - → number of periods of time which occur in one year: k

$$i_{period} = i_a / k$$

Ex.º $i_m = i_a / 12$ will be the monthly interest rate that is proportional to i_a

- ***Equivalent interest rate***: is the interest rate that yields the same accumulated value over same time period (generating an equivalent amount in the end of the period). Example: which will be the monthly interest rate (i_m) that is equivalent to an annual interest rate i_a? It is the one that will yields the same accumulated capital at the end of one year (or 12 months)

$$(1+i_m)^{12} = (1+i_a) \quad \text{or} \quad i_m = (1+i_a)^{1/12} - 1$$

Generalizing:

$$i_{period} = (1+i_a)^{1/k} - 1$$

where "period" is the number of periods that occur during one year (if we are thinking about months → they are 12 in one year).

Rents

- ***Rents (def):*** Series of payments that occur during a certain period of time. They can classified according to 4 criteria:

i) Variability of the terms:

. Constant: All terms are identical
. Variable: Terms vary.

ii) Number of terms:

. Temporary (finite number of terms).
. Perpetual (infinite number of terms).

iii) Period of rent and of interest rate:

. Coincident: The periods coincide.
. Fractioned: Periods do not coincide.

iv) Moment where the fist term occurs:

. Normal: 1st term occurs after one period of time

. Anticipated: 1st term occurs immediately (moment 0)

. Postponed: 1st term occurs after m periods of time (m>1)

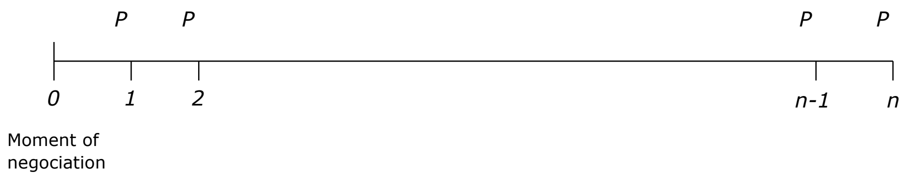
Rents

- ***Annua**lity (def): A constant rent with a <u>finite number of</u> terms)* *where:*

  The rent and the interest rate <u>coincide</u>

  and is <u>normal</u> (1st term occurs after one period of time).

Suppose there is a series of payments of a certain amount "a" every period during n periods of time



Present Value:

- $V_0 = P/(1+i) + P/(1+i)^2 + \dots + P/(1+i)^n$
- $V_0 = \frac{P}{i}\left[1 - \frac{1}{(1+i)^n}\right]$

Rents

- ***Demonstration:***

$V_0 = P/(1+i) + P/(1+i)^2 + \dots + P/(1+i)^n$

$V_0 = P\,[1/(1+i) + (1/(1+i))^2 + \dots + (1/(1+i))^n]$

sum of the terms of a geometric progression with reason 1/(1+i)

*formula for the sum: 1st term * (1-raason ^{n})/1-reason). Substituting:*

$$V_0 = P\,\frac{1}{(1+i)}\,\frac{1-1/(1+i)^n}{1-1/(1+i)}$$

$$V_0 = P\,\frac{1}{(1+i)}\,\frac{1-1/(1+i)^n}{\frac{1+i-1}{1+i}} \longrightarrow V_0 = \frac{P}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

Rents

Capitalized or Future Value of a Rent:

$$V_n = P(1+i)^{n-1} + P(1+i)^{n-2} + \ldots + P(1+i) + P(1+i)^0$$

$$V_n = \frac{P}{i}\left[(1+i)^n - 1\right]$$

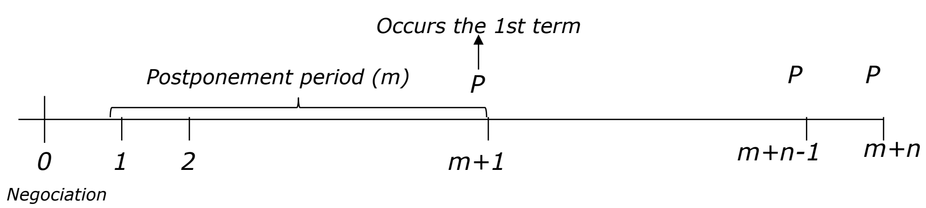
Note: The capitalized value after n periods is the same as the Present Value multiplied by $(1+i)^n$

$$V_n = V_0 (1+i)^n = \frac{P}{i}\left[1 - 1/(1+i)^n\right](1+i)^n$$

$$V_n = \frac{P}{i}\,\frac{\left[(1+i)^n - 1\right]}{(1+i)^n}\,(1+i)^n \Rightarrow V_n = \frac{P}{i}\left[(1+i)^n - 1\right]$$

Postponed Rents

- *Consider an Annuity (constant and finite rent where periods coincide with interest periodicity) where the* ***1st term occurs after m periods***



Present Value (today):

$$V_0 = \frac{P}{i\,(1+i)^m}\left[1 - \frac{1}{(1+i)^n}\right]$$

Postponed Rents

Demonstration:

$$V_0 = P/(1+i)^{m+1} + P/(1+i)^{m+2} + \dots + P/(1+i)^{m+n}$$

$$V_0 = \frac{P}{(1+i)^m} \underbrace{[1/(1+i) + (1/(1+i))^2 + \dots + (1/(1+i))^n]}$$

sum of the terms of a geometrical progression with reason 1/(1+i) with n terms.

Formula for the sum: : SUM = 1st term $\frac{1 - reason^n}{1 - reason}$

Substituting:

$$V_0 = \frac{P}{(1+i)^m} \frac{1}{(1+i)} \frac{1-1/(1+i)^n}{1-1/(1+i)}$$

$$V_0 = \frac{P}{(1+i)^m} \frac{1}{(1+i)} \frac{1-1/(1+i)^n}{\frac{1+i-1}{1+i}} \longrightarrow V_0 = \frac{P}{i(1+i)^m}\left[1 - \frac{1}{(1+i)^n}\right]$$

Postponed Rents

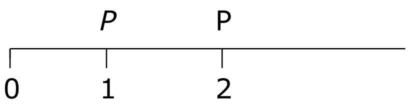
Capitalized (or future) Value of an Annuity that is postponed m periods:

$$V_n = P(1+i)^{m+n-(m+1)=n-1} + P(1+i)^{n-2} + \dots + P(1+i)^0$$

$$V_n = \frac{P}{i(1+i)^m}[(1+i)^n - 1]$$

Perpetuities

- ***Perpetuity*** *(Def): normal rent with a constant and an infinite number of terms) where the periodicity of the rent and of the interest rate coincide;*



Present Value: $V_0 = \lim_{n \to \infty} \frac{P}{i} \left[1 - \frac{1}{(1+i)^n}\right]$

$$V_0 = \frac{P}{i}.$$

Variable Rents (growing terms)

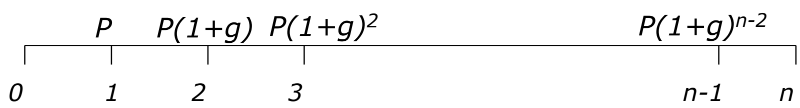
Variable Rents *where:*

number of terms is finite

rent and interest rate periodicity coincide

normal (1st term occurs after one period).

Particular case: The payments grow at a certain growth rate g :



Present Value (today):

$$V_0 = \frac{P}{i-g} \left[1 - \left(\frac{1+g}{1+i}\right)^n\right]$$

Variable Rents (growing terms)

Demonstration:

$$V_0 = P/(1+i) + P(1+g)/(1+i)^2 + \ldots + P(1+g)^{n-1}/(1+i)^n$$

$$V_0 = \frac{P}{1+i}\left[1 + \frac{1+g}{1+i} + \frac{(1+g)^2}{(1+i)^2} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^{n-1}}\right]$$

sum of the terms of a geometrical progression with reason (1+g)/(1+i) with n terms.

Formula for the sum: $SUM = 1^{st}\ term\ \frac{1 - reason^n}{1 - reason}$

Substituting:

$$V_0 = \frac{P}{(1+i)} \cdot \frac{1-[(1+g)/(1+i)]^n}{1-(1+g)/(1+i)}$$

$$V_0 = \frac{P}{(1+i)} \cdot \frac{1-[(1+g)/(1+i)]^n}{\frac{1+i-1-g}{1+i}} \longrightarrow V_0 = \frac{P}{i-g}\left[1 - \left(\frac{1+g}{1+i}\right)^n\right]$$

note: This tends to the formula for the annuity when g=0 (constant rent)

Variable Rents (growing terms)

Capitalized Value (or future value, or final value):

$$V_n = \frac{P}{i-g}\left[(1+i)^n - (1+g)^n\right]$$